Periodic solutions of nonlinear equations obtained by linear superposition

2002 J. Phys. A: Math. Gen. 35 10085

(http://iopscience.iop.org/0305-4470/35/47/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 10085–10100 PII: S0305-4470(02)38002-8

Periodic solutions of nonlinear equations obtained by linear superposition

Fred Cooper[1], **Avinash Khare**[2,3] **and Uday Sukhatme**[2]

[1] Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
[2] Department of Physics, University of Illinois at Chicago, Chicago, IL 60607-7059, USA

Received 12 June 2002, in final form 30 August 2002
Published 12 November 2002
Online at stacks.iop.org/JPhysA/35/10085

Abstract

We show that a type of linear superposition principle works for several nonlinear differential equations. Using this approach, we find periodic solutions of the Kadomtsev–Petviashvili equation, the nonlinear Schrödinger equation, the $\lambda\phi^4$ model, the sine-Gordon equation and the Boussinesq equation by making appropriate linear superpositions of known periodic solutions. This unusual procedure for generating solutions of nonlinear differential equations is successful as a consequence of some powerful, recently discovered, cyclic identities satisfied by the Jacobi elliptic functions.

PACS numbers: 02.30.Jr, 02.30.Gp, 05.45.Yv

1. Introduction

The fact that Jacobi elliptic functions arise naturally as travelling wave solutions of many nonlinear systems has been known for quite some time (see, for example, [1]). Although for the solitary wave solutions of these nonlinear equations, there is no superposition principle (except when the solitary waves are far apart), for the periodic solutions the situation turns out to be quite different. It has recently been shown [2] that certain specific linear combinations of known periodic solutions of the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations as well as $\lambda\phi^4$ theory, also satisfy these equations. This unexpected result is a consequence of some remarkable, recently established, identities involving Jacobi elliptic functions [3]. Basically, the identities take the cross terms generated by the nonlinear terms in the KdV and mKdV equations and convert them into a manageable form. The purpose of this paper is to show that such a procedure also works for other well-known nonlinear equations, namely the Kadomtsev–Petviashvili (KP) equation, the nonlinear Schrödinger equation (NLSE), the $\lambda\phi^4$ model, the sine-Gordon equation and the Boussinesq equation. It should be noted that the above list includes both the integrable and nonintegrable systems. These equations are of interest in several areas of physics. The NLSE governs the propagation

[3] Permanent address: Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, Orissa, India.

of an electromagnetic wave in a glass fibre, or the spatial evolution of an electromagnetic field in a planar waveguide. Temporal solitons described by the NLSE were first observed in 1980 [4], and the first confirmation and studies of spatial solitons in planar waveguides were reported in 1988 [5, 6]. Similarly, the $\lambda\phi^4$ and the sine-Gordon equations arise in several areas of condensed matter physics.

At this point one might wonder if the idea of superposition was used earlier to obtain periodic solutions of nonlinear equations. So far as we are aware, the answer to this question is no. However, mention might be made of some recent work [7] where Jacobi elliptic function expansion methods have been used to construct some exact periodic solutions of several nonlinear wave equations. We might add here that similar methods had also been proposed earlier [8] to obtain the shock and solitary wave solutions of nonlinear equations.

2. The Kadomtsev–Petviashvili (KP) equation

The KP equation is a two-dimensional generalization of the KdV equation and is given by

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{1}$$

Properties of the KP equation are discussed in many texts [1]. In particular, the simplest, periodic, cnoidal travelling wave solution is

$$u_1(x, y, t) = -2\alpha^2 \,\mathrm{dn}^2(\xi_1, m) + \beta\alpha^2 \qquad \xi_1 \equiv \alpha(x + \gamma\alpha y - b_1\alpha^2 t) \tag{2}$$

where α, γ, m and β are constants, and the 'velocity' b_1 is given by

$$b_1 = 8 - 4m - 6\beta + 3\gamma^2. \tag{3}$$

In this paper, for Jacobi elliptic functions, we use the standard notation $\mathrm{dn}(\xi, m)$, $\mathrm{sn}(\xi, m)$, $\mathrm{cn}(\xi, m)$, where m is the elliptic modulus parameter ($0 \leqslant m \leqslant 1$). Solution (2) remains unchanged when x is increased by $2K(m)/\alpha$, where $K(m)$ is the complete elliptic integral of the first kind [9]. In the limiting case $m = 1$ (and $\beta = 0$), one recovers the familiar single soliton form $-2\alpha^2 \mathrm{sech}^2(\alpha(x + \gamma\alpha y - b_1\alpha^2 t))$.

We will make suitable linear combinations of solution (2) and show that the result is also a periodic solution of the KP equation. Our procedure consists of adding terms of the kind given in (2) but centred at p equally spaced points along the period $2K(m)/\alpha$, where p is any integer. The p-point solution is

$$\begin{gathered} u_p(x, y, t) = -2\alpha^2 \sum_{i=1}^{p} d_i^2 + \beta\alpha^2 \qquad d_i \equiv \mathrm{dn}\left[\xi_p + \frac{2(i-1)K(m)}{p}, m\right] \\ \xi_p \equiv \alpha(x + \gamma\alpha y - b_p\alpha^2 t). \end{gathered} \tag{4}$$

Clearly, $p = 1$ is the original solution, but for any other p, we have expressions which, as we shall show, also solve the KP equation. For convenience, we define the quantities s_i and c_i in analogy with the quantity d_i defined above:

$$s_i \equiv \mathrm{sn}\left[\xi_p + \frac{2(i-1)K(m)}{p}, m\right] \qquad c_i \equiv \mathrm{cn}\left[\xi_p + \frac{2(i-1)K(m)}{p}, m\right]. \tag{5}$$

The KP equation contains the KdV operator $u_t - 6uu_x + u_{xxx}$. It has been shown in detail in [2] that equation (4) with $\gamma = 0$ is a solution of the KdV equation. The proof is based on the identity

$$\sum_{i<j}^{p} d_i^2 d_j^2 = A_1(p, m) \sum_{i=1}^{p} d_i^2 + A_2(p, m). \tag{6}$$

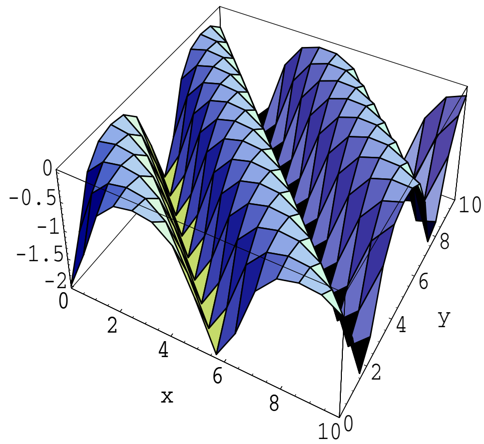

Figure 1. $u_1(x, y, t = 0)$ versus (x, y) for $m = 0.95$, $\alpha = \gamma = 1$, $\beta = 0$.

This is one of many powerful new identities [3] which reduce by 2 (or a larger even number) the degree of cyclic homogeneous polynomials in Jacobi elliptic functions. The constants $A_1(p, m)$ and $A_2(p, m)$ in identity (6) can be evaluated in general by choosing any specific convenient value of the argument ξ of the Jacobi elliptic functions. The results for $A_1(p, m)$ for small values of p are

$$A_1(p = 2, m) = 0 \qquad A_1(p = 3, m) = \frac{-2(m - 1 + q^2)}{1 - q^2}$$
$$A_1(p = 4, m) = -2\sqrt{1 - m} \tag{7}$$

where

$$q \equiv \mathrm{dn}(2K(m)/3, m). \tag{8}$$

The limiting values at $m = 0, 1$ are also particularly simple:

$$A_1(p, m = 0) = -\tfrac{1}{3}(p - 1)(p - 2) \qquad A_1(p, m = 1) = 0. \tag{9}$$

Taking expression (4) and using identity (6), the left-hand side of the KP equation (1) becomes

$$4m\alpha^5\{8 - 4m - 6\beta - b_p + 12A_1(p, m)\}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{i=1}^{p} s_i c_i d_i + 12m\gamma\alpha^4\frac{\mathrm{d}}{\mathrm{d}y}\sum_{i=1}^{p} s_i c_i d_i. \tag{10}$$

Clearly, this vanishes if the velocity is given by

$$b_p = 8 - 4m - 6\beta + 12A_1(p, m) + 3\gamma^2. \tag{11}$$

Thus for this choice of velocity, the KP equation is solved by our p-point expression (4). Effectively, the new solutions of the KP equation and the corresponding solutions of the KdV equation have a difference of $3\gamma^2$ in their velocities b_p. Note that as in the KdV case, the results for b_p can be positive or negative depending on the values of the parameters [2]. The behaviour of u_1 for the parameters $m = 0.95$, $\alpha = \gamma = 1$, $\beta = 0$ is shown in figure 1.

In addition to the solution (2), another well-known periodic solution of the KP equation (1) of period $4K(m)$ is

$$v_1(x, y, t) = \alpha^2[m\,\mathrm{sn}^2(\eta_1, m) \pm \sqrt{m}\,\mathrm{cn}(\eta_1, m)\,\mathrm{dn}(\eta_1, m)] \qquad \eta_1 \equiv \alpha(x + \gamma\alpha y - q_1\alpha^2 t) \tag{12}$$

with velocity $q_1 = (-1 - m + 3\gamma^2)$.

Starting from this solution, we can again obtain, by superposition, periodic solutions of the KP equation of period $4K(m)/p$ in case p is an odd integer. The general p-point solution is given by

$$v_p(x, y, t) = \alpha^2 \sum_{i=1}^{p} \left[m\tilde{s}_i^2 \pm \sqrt{m}\,\tilde{c}_i\tilde{d}_i\right] \qquad p \text{ odd} \tag{13}$$

where we define

$$\begin{aligned} \tilde{s}_i &\equiv \text{sn}\left[\eta_p + \frac{4(i-1)K(m)}{p}, m\right] \qquad \tilde{c}_i \equiv \text{cn}\left[\eta_p + \frac{4(i-1)K(m)}{p}, m\right] \\ \tilde{d}_i &\equiv \text{dn}\left[\eta_p + \frac{4(i-1)K(m)}{p}, m\right]. \end{aligned} \tag{14}$$

As has been shown in detail in [2], equation (13) with $\gamma = 0$ is a solution of the KdV equation with velocity

$$q_p = -(1+m) - 6[B_1(p, m) - C_1(p, m)] \tag{15}$$

where the quantities $B_1(p, m)$ and $C_1(p, m)$ come from the following identities:

$$m\sum_{i<j}^{p} \tilde{s}_i\tilde{s}_j = B_1(p, m) \qquad m\sum_{i<j<k}^{p} \tilde{s}_i\tilde{s}_j\tilde{s}_k = C_1(p, m). \tag{16}$$

It is easily checked that even in the KP case ($\gamma \neq 0$), equation (13) is an exact solution, the only difference being that the velocity in the KP case is larger by $3\gamma^2$. As an illustration, for the $p = 3$ case, it is easily shown that [3]

$$B_1(3, m) = -(1 - q^2) \qquad C_1(3, m) = -m/(1 - q^2) \tag{17}$$

so that the velocity of the KP soliton is given by

$$q_3 = -1 - m + 6(1 - q^2) - \frac{6m}{1 - q^2} + 3\gamma^2 \tag{18}$$

where q has been defined in equation (8). Note that for p even, we do not obtain any new solutions. To illustrate our results, in figure 2 we plot $v_3(x, y)$ at time $t = 0$ for the choice $\alpha = \gamma = 1$ and $m = 0.95$.

3. The nonlinear Schrödinger equation

The NLSE with both attractive and repulsive nonlinearity has found many physical applications in several diverse areas including fibre optics, Bose–Einstein condensates and waveguides [10].

3.1. Case I: attractive nonlinearity

The NLSE with attractive nonlinearity is given by ($\hbar = 2m = 1$)

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0 \tag{19}$$

where without any loss of generality we have fixed the coefficient of the nonlinear term to be unity. As usual, one starts with the ansatz [1]

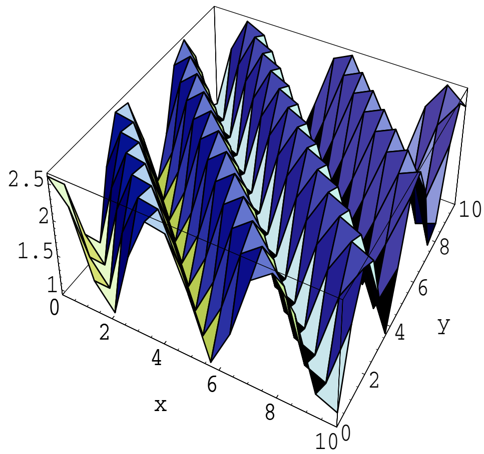

Figure 2. $v_3(x, y, t = 0)$ versus (x, y) for $m = 0.95$, $\alpha = \gamma = 1$.

$$u(x,t) = r(\xi)\,\mathrm{e}^{\mathrm{i}(\theta(\xi)+nt)} \qquad \xi \equiv x - vt \tag{20}$$

which on substituting in equation (19) yields

$$\theta'(\xi) = \frac{1}{2}\left(v + \frac{A}{r^2}\right) \tag{21}$$

$$r'^2(\xi) = -\frac{r^4}{2} + \left(n - \frac{v^2}{4}\right)r^2 - \frac{B}{2} - \frac{A^2}{4r^2} \tag{22}$$

where prime denotes a derivative with respect to the argument (ξ) and A and B are constants of integration. Thus, the whole problem reduces to finding the solutions of equation (22), after which θ is easily obtained by using equation (21) and performing one integration.

The well-known soliton solution of equation (22) is

$$r(\xi) = \sqrt{2}\,\mathrm{sech}(\xi) \qquad \theta = \frac{v\xi}{2} \qquad A = B = 0 \qquad v^2 = 4(n-1) \tag{23}$$

which is valid only for $n \geqslant 1$. It may be noted that a somewhat more general solution with arbitrary amplitude α is easily obtained, since if $u(x,t)$ is a solution of the NLSE, then $\alpha u(\alpha x, \alpha^2 t)$ is also a solution of the same equation.

The two simplest, periodic, cnoidal travelling wave solutions of equation (22) are $(\xi_1 = x - v_1 t)$

$$r_1(\xi) = \sqrt{2}\,\mathrm{dn}(\xi_1) \quad \theta_1 = \frac{v_1\xi_1}{2} \quad A = 0 \quad B = 4(1-m) \quad v_1^2 = 4(n+m-2) \tag{24}$$

$$r_1(\xi) = \sqrt{2m}\,\mathrm{cn}(\xi_1) \quad \theta_1 = \frac{v_1\xi_1}{2} \quad A = 0 \quad B = -4m(1-m) \quad v_1^2 = 4(n+1-2m). \tag{25}$$

In the limiting case $m = 1$, one recovers the familiar soliton solution (23).

We shall now show that suitable linear combinations of solutions (24) and (25) are also solutions of equation (22). Consider first solution (24). Our solutions consist of adding terms of the kind given in this equation but centred at p equally spaced points along the period $2K(m)$, where p is any integer. The p-point solution is

$$r_p(x,t) = \sqrt{2}\sum_{i=1}^{p} d_i \qquad d_i \equiv \mathrm{dn}\left[\xi_p + \frac{2(i-1)K(m)}{p}, m\right] \qquad \xi_p \equiv (x - v_p t). \tag{26}$$

Clearly, $p = 1$ is the original solution, but for any other p, we have solutions of period $2K(m)/p$.

In order to verify that expression (26) is indeed a solution of equation (22), one needs the identities

$$\left(\sum_{i=1}^{p} d_i\right)^2 = \sum_{i=1}^{p} d_i^2 + A(p,m)$$
$$\left(\sum_{i=1}^{p} d_i\right)^4 = \sum_{i=1}^{p} d_i^4 + C(p,m)\sum_{i=1}^{p} d_i^2 + D(p,m) \tag{27}$$
$$m^2 \sum_{i<j}^{p} s_i c_i s_j c_j = E(p,m)\sum_{i=1}^{p} d_i^2 + F(p,m)$$

which can easily be established by following the procedure discussed in [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + m - 2 - C(p,m) - 2E(p,m)]. \tag{28}$$

Some explicitly computed values of the constants $C(p,m)$ and $E(p,m)$ are

$$C(2,m) = 4E(2,m) = 4\sqrt{1-m} \qquad C(3,m) = 4E(3,m) = \frac{8mq}{1-q^2}$$
$$C(4,m) = 4E(4,m) = 4\tilde{t}(2+\tilde{t}+2\tilde{t}^2) \tag{29}$$

where q is given by equation (8) and $\tilde{t}$ is given by

$$\tilde{t} \equiv (1-m)^{1/4}. \tag{30}$$

On the other hand, for any p at $m = 0$, $C(p,0) = 4E(p,0) = \frac{4(p^2-1)}{3}$ and at $m = 1$, $C(p,1) = E(p,1) = 0$. It then follows from equation (28) that the solution r_p as given by equation (26) is valid only if $n \geqslant 2p^2$, and in this case v^2 changes from $4(n-2p^2)$ to $4(n-1)$ as m goes from 0 to 1.

For odd p, using solution (25), we obtain the following solution of the NLSE (22) by linear superposition:

$$r_p(x,t) = \sqrt{2m}\sum_{i=1}^{p} \tilde{c}_i \qquad \tilde{c}_i \equiv \mathrm{cn}\left[\eta_p + \frac{4(i-1)K(m)}{p}, m\right] \qquad \eta_p \equiv (x - v_p t). \tag{31}$$

In order to verify that (31) is indeed a solution to the NLSE (22) one needs the identities

$$\left(\sum_{i=1}^{p} \tilde{c}_i\right)^2 = \sum_{i=1}^{p} \tilde{c}_i^2 + G(p,m)$$
$$\left(\sum_{i=1}^{p} \tilde{c}_i\right)^4 = \sum_{i=1}^{p} \tilde{c}_i^4 + H(p,m)\sum_{i=1}^{p} \tilde{c}_i^2 + I(p,m) \tag{32}$$
$$m^2 \sum_{i<j}^{p} \tilde{s}_i \tilde{d}_i \tilde{s}_j \tilde{d}_j = J(p,m)\sum_{i=1}^{p} \tilde{c}_i^2 + K(p,m)$$

which can be established following the procedure discussed in [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + 1 - 2m - mH(p,m) - 2J(p,m)]. \tag{33}$$

Some explicitly computed values of the constants $H(p,m)$ and $J(p,m)$ are

$$mH(3,m) = 4J(3,m) = -4q\left[\frac{q+2}{(1+q)^2} + \frac{q}{1-q^2}\right] \tag{34}$$

where q is given by equation (8). Thus v^2 varies from $4(n+9)$ to $4(n-1)$ as the elliptic modulus parameter m changes from 0 to 1.

On the other hand, for even p, we have obtained the following solution of the NLSE (22):

$$r_p(x,t) = \sqrt{2}\sum_{i\text{ odd}}^{p}[d_i - d_{i+1}]. \tag{35}$$

In order to verify that equation (35) is a solution of the NLSE (22) one needs the identities

$$\begin{aligned}\left(\sum_{i\text{ odd}}^{p}[d_i - d_{i+1}]\right)^2 &= \sum_{i=1}^{p} d_i^2 + P(p,m)\\ \left(\sum_{i\text{ odd}}^{p}[d_i - d_{i+1}]\right)^4 &= \sum_{i=1}^{p} d_i^4 + L(p,m)\sum_{i=1}^{p} d_i^2 + M(p,m)\\ m^2\left[\sum_{i+j\text{ even}}^{p} s_i c_i s_j c_j - \sum_{i+j\text{ odd}}^{p} s_i c_i s_j c_j\right] &= N(p,m)\sum_{i=1}^{p} d_i^2 + Q(p,m)\end{aligned} \tag{36}$$

which can be established following the procedure discussed in [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + m - 2 - L(p,m) - 2N(p,m)]. \tag{37}$$

Some explicitly computed values of the constants $L(p,m)$ and $N(p,m)$ are

$$L(2,m) = 4N(2,m) = -4\sqrt{1-m} \qquad L(4,m) = 4N(4,m) = -4\tilde{t}(2 - \tilde{t} + 2\tilde{t}^2) \tag{38}$$

where $\tilde{t}$ is as given by equation (30). Thus for $p = 2\,[4]$, v^2 varies from $4(n+4)[4(n+16)]$ to $4(n-1)$ as m changes from 0 to 1. Generalizing the results in equations (31) or (35), one finds that v^2 varies from $4(n+p^2)$ to $4(n-1)$ as m changes from 0 to 1.

3.2. Case II: repulsive nonlinearity

The NLSE with repulsive nonlinearity is given by

$$\mathrm{i}u_t + u_{xx} - u|u|^2 = 0. \tag{39}$$

We again start with the ansatz given by equation (20) and on following the same steps as given in equations (20) to (22) it is easily seen that the θ equation (equation (21)) is the same as before while the r equation is almost the same except for the sign of the r^4 term. In particular, the r equation is now given by

$$r'^2(\xi) = \frac{r^4}{2} + \left(n - \frac{v^2}{4}\right)r^2 - \frac{B}{2} - \frac{A^2}{4r^2}. \tag{40}$$

The well-known soliton solution to this equation is

$$r(\xi) = \sqrt{2}\tanh(\xi) \qquad \theta = \frac{v\xi}{2} \qquad A = 0 \qquad B = -4 \qquad v^2 = 4(n+2). \tag{41}$$

The simplest, periodic, cnoidal travelling wave solution to equation (40) is

$$r_1(\xi) = \sqrt{2m}\,\mathrm{sn}(\xi_1) \qquad \theta_1 = \frac{v_1\xi_1}{2} \qquad A = 0 \qquad B = -4m \qquad v_1^2 = 4(n+1+m). \tag{42}$$

For odd p, using solution (42), we obtain the following solutions of the NLSE (22) by linear superposition:

$$r_p(x,t) = \sqrt{2m}\sum_{i=1}^{p}\tilde{s}_i \qquad \tilde{s}_i \equiv \mathrm{sn}\left[\eta_p + \frac{4(i-1)K(m)}{p}, m\right] \qquad \eta_p \equiv (x - v_p t). \tag{43}$$

In order to verify that equation (43) is a solution to the NLSE (40) one needs the identities

$$\begin{aligned}\left(\sum_{i=1}^{p}\tilde{s}_i\right)^2 &= \sum_{i=1}^{p}\tilde{s}_i^2 + R(p,m)\\ \left(\sum_{i=1}^{p}\tilde{s}_i\right)^4 &= \sum_{i=1}^{p}\tilde{s}_i^4 + S(p,m)\sum_{i=1}^{p}\tilde{s}_i^2 + T(p,m)\\ \sum_{i<j}^{p}\tilde{c}_i\tilde{d}_i\tilde{c}_j\tilde{d}_j &= U(p,m)\sum_{i=1}^{p}\tilde{s}_i^2 + Y(p,m)\end{aligned} \tag{44}$$

which can be established by following [3]. The general expression for the velocity v_p is

$$v_p^2 = 4[n + 1 + m + mS(p,m) - 2U(p,m)]. \tag{45}$$

Some explicitly computed values of the constants $H(p,m)$ and $J(p,m)$ are

$$mS(3,m) = -4U(3,m) = 4m\left[\frac{1}{1-q^2} - \frac{1-q^2}{m}\right] \tag{46}$$

where q is given by equation (8). Thus v^2 changes from $4(n+9)$ to $4(n+2)$ as m changes from 0 to 1.

For even integer p, the linear superposition of elementary solutions does not work. However, remarkably enough we find that the products of elementary solutions are also solutions. For example, the solution for $p = 2$ is

$$r_2(x,t) = \sqrt{2}\, m s_1 s_2 \tag{47}$$

and the corresponding velocity is given by $v_2^2 = 4(n+4-2m)$. Generalization to higher even values of p is straightforward.

3.3. Solutions with $A \neq 0$

It may be noted that since $A = 0$ for all the solutions discussed so far, the expressions for θ were rather trivial. One way of obtaining a solution with $A \neq 0$ is to start with the ansatz

$$r^2(\xi) = 2\,\mathrm{dn}^2(\xi) + \alpha \tag{48}$$

where α is a constant. It is easily checked that (48) is a solution to the NLSE (22) provided

$$\begin{aligned}\alpha &= \frac{2}{3}\left(n - 2 + m - \frac{c^2}{4}\right) \qquad A^2 = 4\alpha\left[\alpha^2 - \left(n - \frac{c^2}{4}\right)\alpha + 2(1-m)\right]\\ B &= -3\alpha^2 + 4\left(n - \frac{c^2}{4}\right)\alpha - 4(1-m).\end{aligned} \tag{49}$$

Starting from the solution (48) we can obtain a class of solutions by an appropriate linear superposition. For example, the 2-point solution is

$$r^2(\xi) = 2\left(d_1^2 + d_2^2\right) + \alpha. \tag{50}$$

It is easily checked that this is indeed a solution provided

$$\alpha = \frac{2}{3}\left(n - 2 + m - \frac{c^2}{4}\right) \qquad B = -3\alpha^2 + 4\left(n - \frac{c^2}{4}\right)\alpha - 16(1-m)$$
$$A^2 = 4\left[\alpha^3 - \left(n - \frac{c^2}{4}\right)\alpha^2 + 8(1-m)\alpha + 8(1-m)\left(n + 2 - m - \frac{c^2}{4}\right)\right]. \tag{51}$$

Generalization to arbitrary p is straightforward.

4. The $\lambda\phi^4$ model

The kink (domain wall) solutions to the $\lambda\phi^4$ field theory in (1 + 1) dimensions

$$\phi_{xx} - \phi_{tt} = \lambda\phi(\phi^2 - a^2) \tag{52}$$

have been widely discussed in the literature [11]. The famous static kink solution is

$$\phi(x) = a\tanh(\sqrt{\lambda/2}\,ax) \tag{53}$$

from which the time-dependent solution

$$\phi(x,t) = a\tanh\left[\sqrt{\frac{\lambda}{2(1-v^2)}}a(x - vt)\right] \tag{54}$$

is immediately obtained by Lorentz boosting. Therefore, to begin with, we shall discuss only the static periodic kink solutions in case $v^2 < 1$ (and $\lambda > 1$). Later, we shall discuss time-dependent solutions with $v^2 > 1$ (or $\lambda < 1$).

4.1. Static periodic kink solutions

It is well known that the static periodic kink solution to the field equation (52) is

$$\phi_1(x) = \sqrt{\frac{2m}{1+m}}a\,\mathrm{sn}(\eta_1, m) \qquad \eta_1 \equiv \sqrt{\frac{\lambda}{1+m}}ax. \tag{55}$$

For any odd integer p, we find the following static kink solutions of the $\lambda\phi^4$ field theory by a specific linear superposition of the basic solution (55):

$$\phi_p(x) = \sqrt{2m}\,\alpha a\sum_{i=1}^{p}\tilde{s}_i \qquad p \text{ odd} \tag{56}$$

where $\tilde{s}_i, \tilde{c}_i, \tilde{d}_i$ are as given in equation (14) with $\eta_p \equiv \sqrt{\lambda}\alpha ax$. In order to verify that equation (56) is a static periodic kink solution to the $\lambda\phi^4$ theory field equation (52), one needs the identity [3]

$$\left(\sum_{i=1}^{p}\tilde{s}_i\right)^3 = \sum_{i=1}^{p}\tilde{s}_i^3 + V(p,m)\sum_{i=1}^{p}\tilde{s}_i \tag{57}$$

and then α is given by

$$\alpha = \frac{1}{\sqrt{1 + m + 2mV(p,m)}}. \tag{58}$$

As an illustration, consider $p = 3$. $V(3,m)$ is given by

$$mV(3,m) = 3\left[\frac{m}{1-q^2} - (1-q^2)\right] \tag{59}$$

where q is defined in equation (8). Note that α varies from 1/3 to $1/\sqrt{2}$ as m varies from 0 to 1.

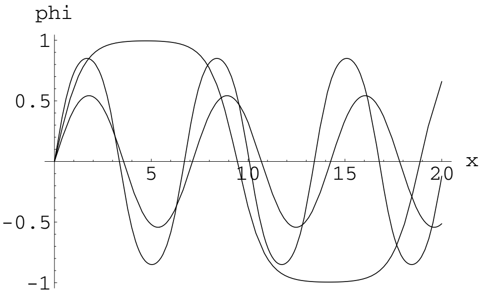

Figure 3. $\phi_1(x)$ (largest amplitude curve), $\phi_2(x)$ and $\phi_3(x)$ (smallest amplitude curve) for $m = 0.98$, $\lambda = a = 1$.

For even integer p, the linear superposition of elementary solutions does not work. However, remarkably enough we find that even in this case, the products of elementary solutions are solutions. For example, the solution for $p = 2$ is

$$\phi_2(x) = \sqrt{2}\, m\alpha a s_1 s_2 \tag{60}$$

where $s_{1,2}$ are as defined in equation (5) with $\xi \equiv \sqrt{\lambda}\,\alpha a x$. Using the identities derived in [3], it is easily shown that (60) is a static periodic kink solution to the field equation (52) provided

$$\alpha = \frac{1}{\sqrt{2(2-m)}}. \tag{61}$$

Note that α varies from $1/2$ to $1/\sqrt{2}$ as m varies from 0 to 1. Generalization to arbitrary even p is straightforward. It is thus clear that for arbitrary integer p, for static periodic kink solutions of $\lambda\phi^4$ theory, α will vary from $1/p$ to $1/\sqrt{2}$ as m varies from 0 to 1. For the values $m = 0.98$ and $\lambda = a = 1$, we plot $\phi_1(x)$ from equation (55), $\phi_2(x)$ from equation (60) and $\phi_3(x)$ from equation (56) in figure 3.

4.2. Periodic time-dependent kink solutions

While in the relativistic field theory context with $\lambda > 0$, equations (53) and (55) are the only solutions of $\lambda\phi^4$ field theory, in the condensed matter physics context, where velocity v can exceed velocity of sound (optical modes), or for relativistic case with $\lambda < 0$, one also has another soliton solution given by

$$\phi(x,t) = \sqrt{2}\, a\, \mathrm{sech}(\beta(x - vt)) \qquad \beta = a\sqrt{\frac{\lambda}{(v^2 - 1)}} \tag{62}$$

which is real if either $\lambda < 0$ (and $v^2 < 1$) or $v^2 > 1$ (and $\lambda > 0$).

The corresponding periodic soliton solutions to equation (52) are well known and given by [12]

$$\phi_1(\xi) = \sqrt{\frac{2}{2-m}}\, a\, \mathrm{dn}(\xi_1, m) \qquad \xi_1 = a\sqrt{\frac{\lambda}{(2-m)\left(v_1^2 - 1\right)}}(x - vt) \tag{63}$$

$$\phi_1(\eta) = \sqrt{\frac{2m}{2m-1}}\, a\, \mathrm{cn}(\eta_1, m) \qquad \eta_1 = a\sqrt{\frac{\lambda}{(2m-1)\left(v_1^2 - 1\right)}}(x - vt). \tag{64}$$

Note that solution (64) is valid only for $1/2 < m < 1$ and both solutions (63) and (64) are only valid for $v^2 > 1$ (and $\lambda > 0$) or $\lambda < 0$ (and $v^2 < 1$).

Appropriate linear superpositions of solutions (63) and (64) are also periodic time-dependent kink solutions. For example, by using the linear superposition of solutions (63), we have the following solution to the field equation (52), which is valid for any integer p:

$$\phi_p(\xi) = \sqrt{2}\, a\alpha \sum_{i=1}^{p} d_i \tag{65}$$

where d_i is as defined in equation (4) with $\xi_p = a\sqrt{\frac{\lambda}{v_p^2-1}}\alpha(x - v_p t)$. In order to prove that this is a solution, one needs the identity

$$\left(\sum_{i=1}^{p} d_i\right)^3 = \sum_{i}^{p} d_i^3 + W(p,m)\sum_{i} d_i \tag{66}$$

which is easily proved following [3]. Using this identity one finds that (65) is a solution provided

$$\alpha^2 = \frac{1}{[2 - m + W(p,m)]}. \tag{67}$$

Some explicitly computed values of $W(p,m)$ are

$$W(2,m) = 3\sqrt{1-m} \qquad W(3,m) = \frac{6mq}{1-q^2} \qquad W(4,m) = 3\tilde{t}[2 + \tilde{t} + 2\tilde{t}^2] \tag{68}$$

where $\tilde{t}$ is as defined in equation (30) and q is given by equation (8). Further, $W(p,0) = p^2 - 1$, while $W(p,1) = 0$.

For any odd integer p, we also have the following solution to the field equation (52) by linear superposition

$$\phi(\eta) = \sqrt{2m}\, a\alpha \sum_{i=1}^{p} \tilde{c}_i \tag{69}$$

where $\tilde{c}_i$ is as defined in equation (14) with $\eta_p = a\sqrt{\frac{\lambda}{v_p^2-1}}\alpha(x - v_p t)$. In order to prove that this is a solution, one needs the identity [3]

$$\left(\sum_{i=1}^{p} \tilde{c}_i\right)^3 = \sum_{i}^{p} \tilde{c}_i^3 + X(p,m)\sum_{i=1}^{p} \tilde{c}_i. \tag{70}$$

Using this identity one finds that (69) is a solution provided

$$\alpha^2 = \frac{1}{[2m - 1 + 2mX(p,m)]}. \tag{71}$$

For example, one can check [3] that

$$X(3,m) = -6(1 - m + q) + \frac{6q^2}{1-q^2} \tag{72}$$

with q being given by equation (8), so that unlike the $p = 1$ case, this is an acceptable solution for all values of m $(0 \leqslant m \leqslant 1)$.

Similarly, for even integer p, we have solutions of the form

$$\phi(\xi) = \sqrt{2}\, a\alpha \sum_{i \text{ odd}}^{p} [d_i - d_{i+1}] \tag{73}$$

where d_i is as defined in equation (5) with $\xi_p = a\sqrt{\frac{\lambda}{v_p^2-1}}\alpha(x - v_p t)$. It is easily checked that this is an exact solution provided

$$\alpha = \frac{1}{[2 - m - 6\sqrt{1-m}]^{1/2}}. \tag{74}$$

This solution is only valid in a very narrow range of values of m corresponding to real values of α.

5. Sine-Gordon field theory

In recent years, both sine-Gordon and $\lambda\phi^4$ field theory have received considerable attention [1, 11]. In particular, sine-Gordon theory is the only relativistically invariant field theory having true soliton solutions. The equation under consideration is

$$\phi_{xx} - \phi_{tt} = \sin\phi. \tag{75}$$

5.1. Static soliton solution by linear superposition

The well-known static one-soliton solution of this equation is given by

$$\phi(x) = 4\tan^{-1} \mathrm{e}^{\pm x}. \tag{76}$$

The corresponding time-dependent solution is easily obtained by Lorentz boosting and hence without any loss of generality we shall restrict our discussion to the static solution only (except when $v^2 > 1$). Solution (76) can also be written in the alternative form

$$\sin\left(\frac{\phi(x)}{2}\right) = \operatorname{sech} x. \tag{77}$$

The two corresponding periodic static soliton solutions are well known and given by [1]

$$\sin\left(\frac{\phi(x)}{2}\right) = \operatorname{dn}(x, m) \tag{78}$$

$$\sin\left(\frac{\phi(x)}{2}\right) = \operatorname{cn}(x/\sqrt{m}, m) \qquad m > 0. \tag{79}$$

For any odd integer p, we obtain the following periodic static soliton solutions by linear superposition:

$$\sin\left(\frac{\phi(x)}{2}\right) = \alpha\sum_{i=1}^{p}\tilde{d}_i \tag{80}$$

where $\tilde{d}_i$ is as defined in equation (14) with $\eta_p \equiv \alpha x$, while α is given by

$$\alpha^2 = \frac{1}{[p + A(p,m) + mR(p,m)]}. \tag{81}$$

Here $A(p,m)$ and $R(p,m)$ are as defined by equations (27) and (44) respectively, and with this choice of α, one obtains

$$\cos\left(\frac{\phi(x)}{2}\right) = \sqrt{m}\,\alpha\sum_{i=1}^{p}\tilde{s}_i. \tag{82}$$

Note that use has been made of the identities [3]

$$\tilde{s}_1(\tilde{c}_2 + \cdots + \tilde{c}_p) + \text{c.p.} = 0 \qquad \tilde{s}_1(\tilde{d}_2 + \cdots + \tilde{d}_p) + \text{c.p.} = 0$$
$$\tilde{d}_1(\tilde{c}_2 + \cdots + \tilde{c}_p) + \text{c.p.} = 0 \tag{83}$$

in proving that (80) is indeed a solution to the field equation (75). For $p = 3$ the values of the constants are

$$A(3, m) = 2q(q + 2) \qquad mR(3, m) = 2(q^2 - 1) \tag{84}$$

where q is given by equation (8), so that $\alpha = 1/(1 + 2q)$ changes from 1/3 to 1 as m varies from 0 to 1.

Another solution valid for any odd integer p is

$$\sin\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{p} \tilde{c}_i \tag{85}$$

where $\tilde{c}_i$ is as defined in equation (14) with $\eta_p \equiv \alpha x/\sqrt{m}$. This solution is strictly valid only if $m > 0$. It is easily checked that this is indeed a solution to the field equation (75) provided

$$\alpha^2 = \frac{1}{[p + G(p, m) + R(p, m)]} \tag{86}$$

where $G(p, m)$ and $R(p, m)$ are as defined by equations (23) and (40), respectively. Note that with this choice of α

$$\cos\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{p} \tilde{s}_i. \tag{87}$$

For $p = 3$, $G(3, m)$ is given by

$$G(3, m) = -\frac{2q(q + 2)}{(1 + q)^2} \tag{88}$$

where $R(3, m)$ is as given by equation (84), so that $\alpha = \frac{1+q}{1-q}$ $(m > 0)$.

Similarly, for $p = 2$ we have the solution

$$\sin\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{2} d_i \tag{89}$$

where d_i is as defined in equation (5) with $\xi_2 \equiv \alpha x$. It is easily checked that this is indeed a solution of the field equation (75) provided

$$\alpha = \frac{1}{1 + \sqrt{1 - m}} \tag{90}$$

so that α varies between 1/2 and 1 as m changes from 0 to 1. Note that with this choice of α

$$\cos\left(\frac{\phi(x)}{2}\right) = m\alpha s_1 s_2. \tag{91}$$

In figure 4, we have plotted $\sin[\phi/2]$ versus x for $p = 1, 2, 3$ corresponding to the right-hand side of equations (78), (89) and (80).

Another solution for $p = 2$ is

$$\sin\left(\frac{\phi(x)}{2}\right) = \alpha(d_1 - d_2) \tag{92}$$

where d_i is as defined in equation (5) with $\xi_2 \equiv \alpha x$. It is easily checked that this is indeed a solution to the field equation (75) provided $0 < m \leqslant 1$ since α given by

$$\alpha = \frac{1}{1 - \sqrt{1 - m}} \tag{93}$$

diverges at $m = 0$.

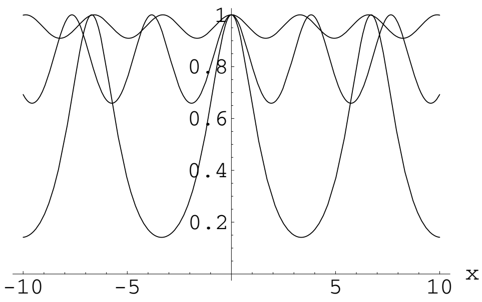

Figure 4. $\sin[\phi(x)/2]$ for $p = 1, 2, 3$. Increasing p decreases the amplitude and increases the frequency.

5.2. *Periodic time-dependent solutions*

As in the $\lambda\phi^4$ field theory case, in this case also we are able to obtain new solutions by linear superposition which are only valid for $v^2 > 1$. In particular, if p is any odd integer, the new periodic time-dependent solution is given by

$$\sin\left(\frac{\phi(x)}{2}\right) = \sqrt{m}\alpha \sum_{i=1}^{p} \tilde{s}_i \tag{94}$$

where $\tilde{s}_i$ is as defined in equation (14) with $\eta_p \equiv \alpha(x - vt)/\sqrt{v^2 - 1}$. It is easily checked that this is indeed a solution of the field equation (75) provided α^2 is again given by equation (81). Note that with this choice of α

$$\cos\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{p} \tilde{d}_i. \tag{95}$$

Another solution for any odd integer p is

$$\sin\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{p} \tilde{s}_i \tag{96}$$

where $\tilde{s}_i$ is as defined in equation (14) with $\eta_p \equiv \alpha(x - vt)/\sqrt{m(v^2 - 1)}$. Thus, this solution is strictly valid only if $m > 0$. It is easily checked that this is indeed a solution of the field equation (75) provided α^2 is as given by equation (86). Note that with this choice of α,

$$\cos\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{p} \tilde{c}_i. \tag{97}$$

Finally, for $p = 2$ we have the solution

$$\sin\left(\frac{\phi(x)}{2}\right) = m\alpha s_1 s_2 \tag{98}$$

where $s_{1,2}$ is as defined in equation (5) with $\xi_2 \equiv \alpha(x - vt)/\sqrt{v^2 - 1}$. It is easily checked that this is indeed a solution of the field equation (75) provided α^2 satisfies equation (89). Note that with this choice of α

$$\cos\left(\frac{\phi(x)}{2}\right) = \alpha \sum_{i=1}^{2} d_i. \tag{99}$$

6. Boussinesq equation

The Boussinesq equation is given by

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0. \tag{100}$$

The periodic one-soliton solution of this equation is known to be

$$u(x,t) = -2\alpha^2\,\mathrm{dn}^2(\alpha[x - vt]) + \beta\alpha^2 \tag{101}$$

where

$$2(4 - 2m - 3\beta)\alpha^2 = v^2 - 1. \tag{102}$$

Thus $v^2 < 1$ if $\beta \geqslant 4/3$ and $v^2 > 1$ if $\beta \leqslant 2/3$. For $2/3 < \beta < 4/3$, v^2 changes sign at some value of m $(0 \leqslant m \leqslant 1)$. Note that in the limit $m \to 1$ and $\beta = 2$, this solution goes over to the one-soliton solution

$$u(x,t) = 2\alpha^2 \tanh^2(\alpha[x - vt]) \qquad \alpha = \sqrt{\frac{1 - v^2}{8}}. \tag{103}$$

Using the above periodic solution, we then obtain several new solutions via linear superposition:

$$u(x,t) = -2\alpha^2 \sum_{i=1}^{p} d_i^2(\alpha[x - vt]) + \beta\alpha^2 \qquad (p = 1, 2, 3, \ldots). \tag{104}$$

It is easy to check that this is an exact solution to the Boussinesq equation (100) provided

$$2[4 - 2m - 3\beta - 6A_1(p,m)]\alpha^2 = v^2 - 1. \tag{105}$$

Here use has been made of the identity (6) with $A_1(p,m)$ given by equations (7) and (10). It may be noted that $A_1(p,m) \leqslant 0$.

7. Summary and conclusions

In this paper, we have made an attempt to see under what conditions a specific kind of linear superposition principle works even for nonlinear equations. We believe that this is an important issue which can help in classifying solutions of nonlinear equations. In view of the remarkable identities satisfied by Jacobi elliptic functions, our linear superposition approach is not only valid for integrable systems such as the KdV and KP equations, but also for nonintegrable systems such as $\lambda\phi^4$ theory. It would indeed be worthwhile to obtain such solutions for other nonlinear systems where elliptic functions play a role in the space of exact solutions.

When is our superposition method expected to work? We have examined several other partial differential equations and find that our method works whenever the periodic one-soliton solution is a sum of integer powers of Jacobi elliptic functions of the form

$$u(x,t) = \sum_i a_i \,\mathrm{sn}^i(\alpha[x - ct], m). \tag{106}$$

Another question which comes to mind is how the solutions obtained in this paper are related to previously known solutions. At first sight, it would appear that our procedure has given new solutions, but a closer investigation reveals that our solutions are expressible in terms of previous solutions via a nonlinear generalization of Landen's formulae which connect Jacobi elliptic functions with two different modulus parameters [13]. The reader is referred to [13] for more details of this unusual and nontrivial generalization.

Acknowledgments

One of us (AK) thanks the Department of Physics at the University of Illinois at Chicago for hospitality. We gratefully acknowledge grant support from the US Department of Energy.

References

[1] Drazin P and Johnson R 1989 *Solitons: an Introduction* (Cambridge: Cambridge University Press)
Ablowitz M and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)
Newell A 1985 *Solitons in Mathematics and Physics* (Philadelphia: SIAM)
[2] Khare A and Sukhatme U 2002 *Phys. Rev. Lett.* **88** 244101
[3] Khare A and Sukhatme U 2002 Cyclic identities involving Jacobi elliptic functions *J. Math. Phys.* **43** 3798
Khare A, Lakshminarayan A and Sukhatme U 2002 Cyclic identities involving Jacobi elliptic functions-II *Preprint* math-ph/0207019
[4] Mollenauer L F, Stolen R H and Gordon J P 1980 *Phys. Rev. Lett.* **45** 1095
[5] Maneuf S and Reynaurd F 1988 *Opt. Commun.* **65** 325
[6] Aitchison J S *et al* 1990 *Opt. Lett.* **15** 471
[7] Liu S, Fu Z, Liu S and Zhao Q 2001 *Phys. Lett.* A **289** 69
Fu Z, Liu S, Liu S and Zhao Q 2001 *Phys. Lett.* A **290** 72
Parkes E J, Duffy B R and Abbott P C 2002 *Phys. Lett.* A **295** 280
[8] Hirota R 1973 *J. Math. Phys.* **14** 810
Otwinowski M, Paul R and Laidlaw W G 1988 *Phys. Lett.* A **128** 483
Wang M L 1995 *Phys. Lett.* A **216** 169
Yan C 1996 *Phys. Lett.* A **224** 77
Parkes E J and Duffy B R 1997 *Phys. Lett.* A **229** 217
Yang L, Liu J and Yang K 2001 *Phys. Lett.* A **278** 267
[9] For the properties of Jacobi elliptic functions, see, for example, Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* (New York: Dover)
Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York: Academic)
[10] See, for example, Carr L D, Clark C W and Reinhardt W P 2000 *Phys. Rev.* A **62** 063610
Carr L D, Clark C W and Reinhardt W P 2000 *Phys. Rev.* A **62** 063611, and references therein
[11] Rajaraman R 1978 *Solitons and Instantons* (Amsterdam: North Holland)
[12] Aubry S 1976 *J. Chem. Phys.* **64** 3392
[13] Khare A and Sukhatme U 2002 A generalization of Landen's quadratic transformation formulas for Jacobi elliptic functions *Preprint* math-ph/0204054
Reinhardt W, Khare A and Sukhatme U 2002 *Comment on 'Linear superposition in nonlinear equations' Preprint* UICHEP-TH/02-5
Cayley A 1876 *An Elementary Treatise on Elliptic Functions* (Cambridge: Cambridge University Press)